OPTIMUM CONTROL OF THE POWER GENERATORS DURING MOTION OF A VARIABLE-MASS BODY WITH ACTIVE JETTISONING OF POWER GENERATORS

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The problem of constructing an optimum law for the decrease in weight of the power generators (with a corresponding decrease in the power of the reactive jet) during the motion of a variable-mass body in a gravitational field was studied earlier in [1-3]. Paper [1] contains an analysis of a stepwise decrease in power. A continuous diminution in power was investigated in [2] and the analysis of the stepwise diminution in power was continued. The results of [3] in which the problem of an optimum decrease in power is solved for constant acceleration due to reactive thrust can be obtained as a particular case from [2].

Below, the problem analyzed in [2] is extended to the case when the jettisoned sections of the power generators can be used partially or completely as propellant to generate thrust* (active jettisoning of generators).

1. Formulation of the variational problem. The system of equations describing the motion of a variable-mass body in a gravitational field and the change in the body weight can be represented as

$$\dot{G}_m = -q_m, \qquad \dot{G}_N = -q_\nu, \qquad \dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{i}\,\frac{\sqrt{(2g/\alpha)\,G_N N\,(q_m + \gamma q_\nu)}}{G_m + G_N + G_n} + \mathbf{R} \tag{1.1}$$

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* The idea of using "excess" parts of the system as propellant was expressed by Tsander [4] in 1909.

It is assumed here that the body weight G is distributed between the store of working material [propellant] G_m, the power generators G_N and the useful load [pay load] G_n, where the weights G_m and G_N are assumed to vary with time t. The weight discharge [consumption] per second $q_m(t) \geqslant 0$ and the γ part $(0 \leqslant \gamma(t) \leqslant \gamma_{max} \leqslant 1)$ of the discharge $q_v(t) \geqslant 0$ are used to generate reactive thrust. The weights G_m, G_N, G_n and the discharges q_m, q_v are referred to the initial body weight.

The reactive jet power $N = qV^2/2$ (where V is the discharge velocity) can vary between zero and a certain maximum value. The maximum value of the power is assumed to be related linearly to the weight of the power source [generators] $N_{max} = G_{N/\alpha}$ (where α is the specific weight of the power source). The power N is considered to be referred to its maximum value so that $0 \leqslant N(t) \leqslant 1$.

The unit vector $\mathbf{i}(t)$ indicates the thrust direction. $\mathbf{r}$ and $\mathbf{v}$ denote the radius-vector and body velocity, $\mathbf{R} = \mathbf{R}(\mathbf{r}, t)$ and g the acceleration due to the gravitational forces at the point $(\mathbf{r}, t)$ and the magnitude of gravity at the earth's surface, respectively. The combination

$$\dot{\sqrt{(2g/\alpha) G_N N (q_m + \gamma q_v)}} / (G_m + G_N + G_n) = a \tag{1.2}$$

is the acceleration due to thrust (thrust divided by the flowing mass). The dot denotes differentiation with respect to time.

Just as in [2], the problem is posed of finding the minimum time of motion T for a given magnitude of useful load G_n. This problem reduces to the following variational problem for system (1.1).

To select from a set of piecewise-continuous, piecewise-smooth functions such controls

$$\begin{gathered} 0 \leqslant N(t) \leqslant 1, \qquad 0 \leqslant \gamma(t) \leqslant \gamma_{\max}, \qquad |\mathbf{i}(t)| \equiv 1 \\ 0 \leqslant q_m(t) < \infty, \qquad 0 \leqslant q_v(t) < \infty \end{gathered} \tag{1.3}$$

as would guarantee minimum time of transition T of system (1.1) for given G_n, γ_{max} and α from the given initial state*

$$G_{m0} + G_{N0} = 1 - G_n \qquad (t_0 = 0) \tag{1.4}$$

* The initial and final conditions for $\mathbf{r}$ and $\mathbf{v}$ are not made specific, it is merely assumed that these conditions satisfy all the requirements imposed for application of the maximum principle [5,6].

into a given final state

$$G_{m1} = 0 \qquad (t_1 = T) \tag{1.5}$$

under the conditions

$$G_m(t) \geqslant 0, \qquad G_N(t) \geqslant 0 \tag{1.6}$$

2. Composition of the optimum control. The formulated variational problem is a problem of the maximum fast-response. The Pontriagin method [5,6] is used in the first step of the solution.

In addition to restrictions (1.3) on the control functions, the problem under consideration contains restrictions (1.6) of the phase coordinates as well. Within the domain ($G_m \geqslant 0$, $G_N \geqslant 0$) the maximum principle is valid according to which the Hamiltonian

$$H = -p_m q_m - p_\nu q_\nu + \mathbf{p}_r \cdot \mathbf{v} + \mathbf{p}_v \cdot \mathbf{R} + p_t + \\ + (\mathbf{p}_v \cdot \mathbf{i}) \sqrt{(2g/\alpha) G_N N (q_m + \gamma q_\nu)} \,/\, (G_m + G_N + G_n) \tag{2.1}$$

$$\left(\dot{p}_m = -\frac{\partial H}{\partial G_m}, \quad \dot{p}_\nu = -\frac{\partial H}{\partial G_N}, \quad \dot{\mathbf{p}}_r = -\frac{\partial H}{\partial \mathbf{r}}, \quad \dot{\mathbf{p}}_v = -\frac{\partial H}{\partial \mathbf{v}}, \quad \dot{p}_t = -\frac{\partial H}{\partial t}\right)$$

at the optimum control should have an absolute maximum in the variables $\mathbf{i}$, N, γ, q_m and q_ν, subject to conditions (1.3).

As regards the boundary $G_N = 0$, it can only be reached at the end of the controlled motion; after this boundary is crossed the thrust vanishes and only motion by means of inertia is possible. The use of the discarded part of the power source as working material makes possible motion along the boundary $G_m = 0$. However, according to [5], for this section the form of the function H and the differential equations for the momenta p are retained, the phase coordinates and the momenta pass through the junction point continuously and the maximum principle seems to be applicable along all the trajectories.

Hence, the optimum control is determined throughout from the condition of the maximum of the Hamiltonian (2.1) in the variables $\mathbf{i}$, N, γ, q_m and q_ν taking account of restrictions (1.3). The unit vector $\mathbf{i}(t)$ in the fourth term of the function H is selected so that the scalar product $\mathbf{p}_v \times \mathbf{i}$, with the non-negative coefficient, would be a maximum

$$\mathbf{p}_v \cdot \mathbf{i} = |\mathbf{p}_v| \tag{2.2}$$

i.e. the thrust vector* must be directed along the vector $\mathbf{p}_v$.

It follows from an analysis of the same term, taking condition (2.2) into account, that the parameter γ and the power N must be as large as possible**

$$\gamma(t) = \gamma_{\max}, \qquad N(t) = 1 \tag{2.3}$$

Afterwards, the part of the function H dependent on the controls q_m and q_v can be written as follows+:

$$H^* = -p_m q_m - p_v q_v - |\mathbf{p}_v| \sqrt{(2g/\alpha) G_N (q_m + \gamma q_v)} / (G_m + G_N + G_n) \tag{2.4}$$

If one or both momenta p_m, p_v are less than or equal to zero, then the optimum value of one or both discharges q_m, q_v becomes infinite, i.e. the condition of piecewise continuity of the controls is violated. If $p_m > 0$ and $p_v > 0$, then the optimum quantities of the discharges are finite and depend on the sign of the function

$$\Delta(t) = p_v(t) - \gamma p_m(t) \tag{2.5}$$

and are determined by the following relations (see Appendix 1):

$$\begin{aligned} q_m &= \frac{(g/\alpha) G_N p_v^2}{2p_m^2 (G_m + G_N + G_n)^2}, \qquad q_v = 0 \qquad \text{for } \Delta > 0 \\ q_m &= 0, \qquad q_v = \frac{\gamma (g/\alpha) G_N p_v^2}{2p_v^2 (G_m + G_N + G_n)^2} \qquad \text{for } \Delta < 0 \end{aligned} \tag{2.6}$$

In the $\Delta(t) = 0$ case, the condition of maximum of the function H^* does not determine either of the discharges q_m and q_v but determines only the sum $(q_m + \gamma q_v)$. The case $\Delta = 0$ is singular in this sense. The optimum values of the discharges are successfully established on the

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* In the singular case $|\mathbf{p}_v(t)| = 0$ it will follow from (2.6) and (2.8) that the discharges q_m and q_v, hence meaning the thrust also, will equal zero. As will be seen from (5.8), $\mathbf{p}_v$ will never vanish except in the degenerate case of entirely passive motion.

** Optimum of the total use of the power for motion with constant weight of the power source was first proved in [7]. When the maximum power of the reactive jet depends on the discharge velocity, the boundary character of the optimum control of the power was established in [8].

\+ Here and henceforth, γ (without a subscript) will be understood to be the maximum value $\gamma_{\max}$.

sections $\Delta(t) = 0$ by using the condition $\dot{\Delta}(t) = 0$ from which it follows that (see Appendix 1)

$$G_m(t) + G_n = (1-2\gamma) G_N(t) \tag{2.7}$$

As is easy to see, this expression has meaning for $\gamma < 0.5$, $G_N < 0.5$. In combination with the condition of maximum of the function $\mathscr{H}$ it yields

$$q_m = (1-2\gamma)\, q_v = \frac{(1-2\gamma)(g/\alpha) G_N p_v^2}{2(1-\gamma) p_m^2 (G_m + G_N + G_n)^2} \quad \text{for } \Delta = 0 \tag{2.8}$$

3. Optimum control of the power source. According to the preceding section, the power which the source communicates to the reactive jet and the fraction γ of the discharge q_v used as the working material must be the greatest possible [see (2.3)]. Moreover

$$q_v = 0 \text{ for } \Delta > 0, \qquad q_m = (1-2\gamma)\, q_v \text{ for } \Delta = 0, \qquad q_m = 0 \text{ for } \Delta < 0$$

[see (2.6), (2.8)]; i.e. equations (1.1) can be written in terms of one of the discharges q_m or q_v on each section of the optimum motion. This discharge is replaced by the new control function a, the acceleration due to thrust (1.2). Then the two vector equations (1.1) determining the motion trajectory will not contain the weight parameters: $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{v}} = a\mathbf{i} + \mathbf{R}$, and equations (1.1) describing the change in the weights G_m and G_N will be expressed in terms of the weight parameters and the square of the acceleration due to thrust

$$\dot{G}_m = -\frac{(G_m + G_N + G_n)^2}{G_N} \frac{\alpha}{2g} a^2, \quad G_N(t) = \text{const} \quad \text{for } \Delta(t) > 0 \tag{3.1}$$

$$\dot{G}_m = -4(1-\gamma)(G_m + G_n) \frac{\alpha}{2g} a^2, \; G_N(t) = \frac{G_m(t) + G_n}{1-2\gamma} \quad \text{for } \begin{cases} \Delta(t) = 0 \\ \gamma < 0.5 \\ G_n < 0.5 \end{cases} \tag{3.2}$$

$$G_m(t) = \text{const}, \qquad \dot{G}_N = -\frac{(G_m + G_N + G_n)^2}{\gamma G_N} \frac{\alpha}{2g} a^2 \quad \text{for } \begin{cases} \Delta(t) < 0 \\ \gamma > 0 \end{cases} \tag{3.3}$$

In order to establish from which of the extrema (3.1) to (3.3) the optimum weight-change law is composed and to determine the alternation of these extrema, it is necessary to investigate the character of the behavior of the function $\Delta(t)$ given by relation (2.5).

At the beginning of the motion

$$\Delta(0) = 1 - \gamma \tag{3.4}$$

since it follows from condition (1.4) on the basis of the general theory [5]* that

$$\text{either } p_{m0} = p_{\nu 0} = -1, \qquad \text{or} \quad p_{m0} = p_{\nu 0} = 1$$

The first possibility drops out because the discharges q_m and q_ν become infinite (see Section 2). Consequently, $\Delta(0) > 0$ [except for the case $\gamma = 1$ when $\Delta(0) = 0$] and an extremum of type (3.1) is realized.

The value of the weight G_{N_1} is not given at the end of the motion, hence, $p_{\nu 1} = 0$. The derivative of the momentum p_m determined by formulas (2.1) with (2.2) and (2.3) taken into account

$$\dot{p}_m = |\, \mathbf{p}_v \,| \sqrt{(2g / \alpha)\, G_N\, (q_m + \gamma q_\nu)} \,/\, (G_m + G_N + G_n)^2 \tag{3.5}$$

is non-negative everywhere. The initial value of this momentum p_{m0} is positive, hence, at the end of the motion

$$\Delta\,(T) = -\,\gamma p_{m1} < 0 \tag{3.6}$$

(except for the case $\gamma = 0$ when $\Delta(T = 0)$ and an extremum of type (3.3) is realized. Thus, the weight G_m, which is constant along this extremum, vanishes according to condition (1.5).

Two versions (for $0 < \gamma < 1$) can occur later.

1. The derivative $\dot{\Delta}$ is always negative, then the optimum law of the weight change consists of extrema of type (3.1) and (3.3).

2. The derivative $\dot{\Delta}$ is first negative, then vanishes at $\Delta = 0$ and remains zero a certain time. After G_m has diminished to zero, the derivative $\dot{\Delta}$ again becomes negative and does not change sign until the end of the motion. In conformity with this, the weight-change law consists of successively joined extrema (3.1), (3.2) and (3.3).

Condition (3.6) is not satisfied for all the remaining versions (see Appendix 2).

In the limiting case $\gamma = 1$ the initial value of the function Δ is zero, hence, the change in weight is determined everywhere by relation (3.3). For the other limiting case $\gamma = 0$, considered in [2], relation (3.3) has no meaning and the optimum weight-change law consists only either of (3.1) or of (3.1) and (3.2).

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* For $t = 0$ the generalized momentum vector must be normal to the hyperplane (1.4). After appropriate normalization it can be considered that $p_{m0} = p_{\nu 0} = \pm 1$.

Let us turn now to the integration of equations (3.1) to (3.3). The variables separate in each of these equations and the quadratures, if the condition of continuous joining of the solutions is taken into account, have the form:

For the case (3.1) + (3.3)

$$-G_{N0}\int_{G_{m0}}^{0}\frac{dG_m}{(G_m+G_{N0}+G_n)^2}-\gamma\int_{G_{N0}}^{G_{N1}}\frac{G_N\,dG_N}{(G_N+G_n)^2}=\Phi \tag{3.7}$$

For the case (3.1) + (3.2) + (3.3)

$$-G_{N0}\int_{G_{m0}}^{G_m^*}\frac{dG_m}{(G_m+G_{N0}+G_n)^2}-\frac{1}{4(1-\gamma)}\int_{G_m^*}^{0}\frac{dG_m}{G_m+G_n}-\gamma\int_{G_N^{**}}^{G_{N1}}\frac{G_N\,dG_N}{(G_N+G_n)^2}=\Phi \tag{3.8}$$

Here

$$\Phi=\frac{\alpha}{2g}\int_0^T a^2dt,\qquad \begin{aligned}G_{m0}&=1-G_{N0}-G_n\\ G_m^*&=(1-2\gamma)\,G_{N0}-G_n\\ G_N^{**}&=G_n/(1-2\gamma)\end{aligned} \tag{3.9}$$

After the integrals (3.7) and (3.8) have been evaluated, a final relation is established between the relative useful load G_n, the initial G_{N0} and the final G_{N1} by means of the values of the weight of the power source, the parameter γ and the magnitude of the functional Φ. Here G_n decreases monotonely as Φ increases, thus permitting the separation of the problems of optimum programming of the acceleration due to thrust and optimum control of the power source.

The optimum program $\mathbf{a}(t) = a\mathbf{i}$ must guarantee minimum time T of displacement ($\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{v}} = \mathbf{a} + \mathbf{R}$) from a given initial and given final state of fixed value of the integral

$$J=\int_0^T a^2dt$$

(or a minimum of the integral J for a fixed time of motion*).

* This problem is considered in [1,7]. The mentioned separation of the general problems is retained even in the presence of restrictions on the acceleration $\mathbf{a}$; for example, $a_{min} \leqslant a(t) \leqslant a_{max}$; $a(t) = a_0$, 0, etc. However, if restrictions are not imposed directly on the acceleration but on the discharge or on the outflow velocity [8], then this property is not retained in the general case. A methodological

Optimum control of the power source is described by relations (2.3), (3.1) to (3.3), (3.7) and (3.8). The initial and final values of the weight of the power source are determined from the condition of maximum of the function $\Phi(G_n, \gamma, G_{N0}, G_{N1})$, given by relations (3.7) and (3.8) by means of the variables G_{N0} and G_{N1} for fixed values of G_n and γ.

For the case (3.1) + (3.3) this procedure leads to the relations

$$G_{N0} = {}^1/_2\,\gamma - G_n + \sqrt{{}^1/_4\,\gamma^2 + (1-\gamma)\,G_n},\quad G_{N1} = 0 \tag{3.10}$$

and for the case (3.1) + (3.2) + (3.3), to the relations

$$G_{N0} = 1/4\,(1-\gamma), \qquad G_{N1} = 0 \tag{3.11}$$

An analysis of the integrated expressions (3.7) and (3.8) in which the optimum values (3.10) and (3.11) of the initial and final weights of the power source [see (3.14), (3.15] have been substituted, permits the establishment of exact ranges of the realization of each type of solution.

The solution (3.1) + (3.3) holds in the ranges

$$\begin{gathered}(1-2\gamma)\,/\,4\,(1-\gamma) \leqslant G_n \leqslant 1, \qquad 0 < \gamma \leqslant 0.5 \\ 0 \leqslant G_n \leqslant 1, \qquad 0.5 \leqslant \gamma < 1\end{gathered} \tag{3.12}$$

and the solution (3.1) + (3.2) + (3.3) in the range

$$0 \leqslant G_n \leqslant (1-2\gamma)\,/\,4\,(1-\gamma), \qquad 0 < \gamma < 0.5 \tag{3.13}$$

The domains (3.12) and (3.13) are shown in Fig. 1. The domain (3.13) is hatched.

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difficulty arises in the search for an optimum magnitude of the constant weight of the power source $G_N(t) \equiv G_{N0}$ in these cases: it is required to find the optimum value of the constant parameter G_{N0} which will enter simultaneously into the right side of the motion equations (1.1) and in the boundary conditions (1.4). In the general theory of the maximum principle [5] a criterion is obtained for the selection of optimum values of the parameters entering only in the right sides of the equations. The mentioned difficulty can be bypassed by introducing the equation $\dot{G}_{N0} = 0$ and by considering G_{N0} not as a parameter but as a phase coordinate. Then the problem with the parameter reduces to a problem without the parameter for whose solution the maximum principle is applicable in its customary formulation.

For $\gamma = 1$ the solution (3.3) is valid in the whole range $0 \leqslant G_n \leqslant 1$ and G_{N0} and G_{N1} are determined by means of formulas (3.10).

For $\gamma = 0$, the solution (3.1) for which

$$G_N(t) \equiv \sqrt{G_n} - G_n$$

is realized in the interval $0.25 \leqslant G_n \leqslant 1$, and we have the solution (3.1) + (3.2) in the interval $0 \leqslant G_n \leqslant 0.25$, where $G_{N0} = 0.25$, $G_{N1} = G_n$.

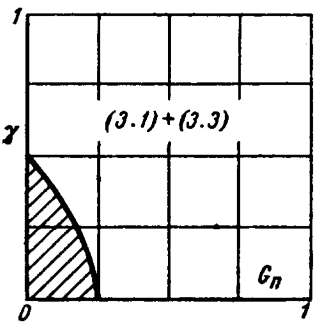

Fig. 1.

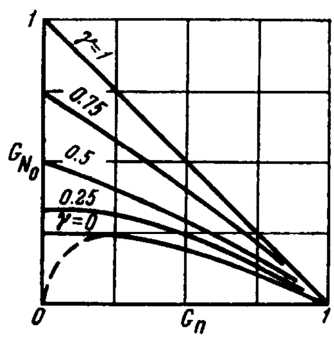

Fig. 2.

The optimum magnitude of the initial weight of the power source G_{N0} as a function of the useful load G_n is presented in Fig. 2 for different values of γ. The curves are constructed by means of formulas (3.10) and (3.11) taking account of the ranges of their realization (3.12) and (3.13). The dashed continuation of the lower curve and the right branch of the curve refer to the case $G_N(t) \equiv \sqrt{(G_n - G_n)}$ [1,7].

The final expression for Φ in terms of G_n and γ is

for $(1-2\gamma)/4(1-\gamma) \leqslant G_n \leqslant 1$, $0 \leqslant \gamma \leqslant 0.5$ and for $0 \leqslant G_n \leqslant 1$, $0.5 \leqslant \gamma \leqslant 1$

$$\Phi = (1-\gamma)(1 - G_n/\varkappa) - \gamma \ln G_n + G_n + \gamma \ln \varkappa - \varkappa \qquad (3.14)$$

$$\left(\varkappa = {}^1/_2\,\gamma + \sqrt{{}^1/_4\,\gamma^2 + (1-\gamma)\,G_n}\right)$$

for $0 \leqslant G_n \leqslant (1-2\gamma)/4(1-\gamma)$, $0 \leqslant \gamma \leqslant 0.5$

$$\Phi = \frac{1-2\gamma}{4(1-\gamma)} - \gamma \ln \frac{1-2\gamma}{2(1-\gamma)} - \frac{1}{4(1-\gamma)} \ln \frac{4G_n(1-\gamma)}{1-2\gamma} \qquad (3.15)$$

The relative useful load G_n as a function of the magnitude of the functional Φ is shown in Fig. 3 for fixed values of γ. The lower solid curve corresponds to the case $\gamma = 0$ analyzed in [2]. This same curve (in the range $0 \leqslant \Phi \leqslant 0.25$) and its dashed continuation (in the range

$0.25 \leqslant \Phi \leqslant 1$) refer to the case of constant weight G_N analyzed in [1,7]. All the curves are shown in the range $0 \leqslant \Phi \leqslant 1$. However, if the function $G_n(\Phi)$ is meaningless outside this range in the case $G_N(t) \equiv \sqrt{G_n} - G_n$, then the useful load is defined on the whole semiaxis $0 \leqslant \Phi < \infty$ for optimally varying weight $G_N(t)$ and for $G_n \ll 1$ the following approximate formulas are valid:

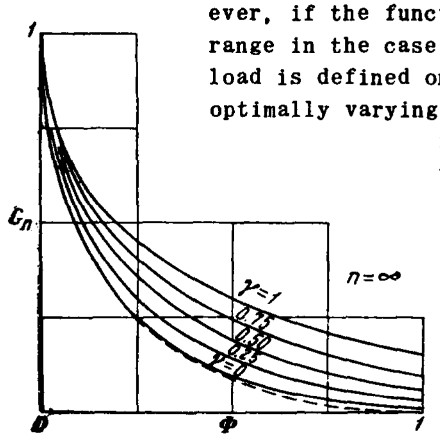

Fig. 3.

for $0 \leqslant \gamma < 0.5$

$$G_n \approx \frac{1-2\gamma}{4(1-\gamma)} \exp\left[-4(1-\gamma)\Phi\right] \quad (3.16)$$

for $0.5 \leqslant \gamma \leqslant 1$

$$G_n \approx \exp(-\Phi/\gamma) \quad (3.17)$$

The dependence of the useful load on the parameter γ is almost linear for small values of $\Phi (0 \leqslant \Phi \leqslant 0.5)$ (Fig. 4) and transforms to the exponential (3.16) and (3.17) for large values.

It follows from a comparison of the upper curve (Fig. 3) ($\gamma = 1$) with the lower solid curve which refers to the case of passive power separation ($\gamma = 0$) that, for the same values of the functional Φ, the use of the discardable sections of the power source as a working material (active jettisoning of generators $\gamma > 0$) essentially increases the useful load: by 1.2-fold for $\Phi = 0.05$, 1.8-fold for $\Phi = 0.25$, more than threefold for $\Phi = 0.5$, sevenfold for $\Phi = 0.75$ and 15-fold for $\Phi = 1$.

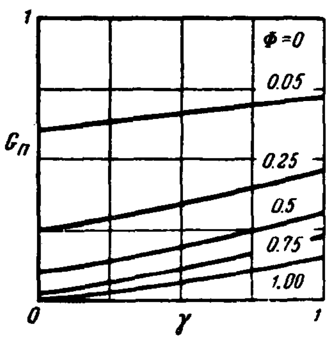

Fig. 4.

However, a comparison of the useful load for the same values of the functional $\Phi = (\alpha/2g)J$ [see (3.9)] yields exaggerated results. It is natural to assume the existence of an increasing dependence of the specific gravity of the power source α on the parameter γ, which will increase the value of Φ for fixed J. Knowledge of the dependence $\alpha(\gamma)$ permits the solution of the problem of selecting the optimum value of γ which would guarantee maximum useful load for a given value of the integral J.

4. Stepwise decrease in the weight of the power source. The case of the continuous decrease in the weight of the power source considered above corresponds to an infinitely large number of sections ($n = \infty$ in Fig. 3). It is interesting to compare this limiting case with the case of a finite number of sections as was done in [2] for $\gamma = 0$.

At time t_j let the jth section of the source be discarded and let the weight of the source remain constant in the interval (t_j, t_{j+1}). The power N and the parameter γ, as before, must be the greatest possible. The change in the body weight is described by (3.1). Integrating this equation by parts from t_j to t_{j+1} ($j = 0, 1, \ldots, n-1$; $t_n = T$) and combining, we obtain

$$\Phi = \sum_{j=0}^{n-1} G_{Nj}^{+}\left(\frac{1}{G_{mj+1}^{-} + G_{Nj}^{+} + G_n} - \frac{1}{G_{mj}^{+} + G_{Nj}^{+} + G_n}\right) \tag{4.1}$$

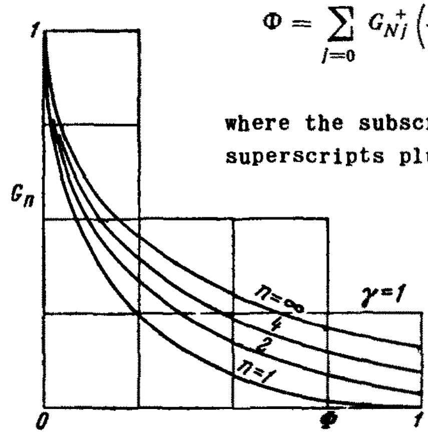

Fig. 5.

where the subscript j indicates the time t_j and the superscripts plus or minus refer to the value of the function to the right or left of the time t_j.

Taking account of the relations on the discontinuities for the store of working material

$$G_{mj}^{+} = G_{mj}^{-} + \gamma\,(G_{Nj}^{-} - G_{Nj}^{+}) \qquad (j = 1, \ldots, n-1)$$

and the condition of constant weight of the power source between the separation times

$$G_{Nj}^{-} = G_{Nj-1}^{+} \qquad (j = 1, \ldots, n)$$

let us rewrite (4.1) as follows:

$$\Phi = \sum_{j=0}^{n-1} G_{Nj}^{+}\left(\frac{1}{G_{mj+1}^{-} + G_{Nj}^{+} + G_n} - \frac{1}{G_{mj}^{-} + \gamma G_{Nj-1}^{+} + (1-\gamma)\,G_{Nj}^{+} + G_n}\right) \tag{4.2}$$

Here

$$G_{N(-1)}^{+} = G_{N0}^{+} = G_{N0}, \qquad G_{m0}^{-} = G_{m0}, \qquad G_{mn}^{-} = G_m(T) = 0$$

By evaluating the partial derivative $\partial\Phi/\partial G_n$ it can be seen that the function $\Phi(G_n)$ is monotonically decreasing. Hence, the problem of seeking the maximum G_n for a given value of Φ can be reduced to finding the values

$$G_{Nj}^{+}, \quad G_{mj}^{-} \qquad (j = 0\ 1, \ldots, n-1) \tag{4.3}$$

which would guarantee a maximum of the function Φ for a fixed value of G_n and would satisfy the conditions

$$G_{m0} + G_{N0} = 1 - G_n, \qquad G_{Nj}^{+} \leqslant G_{Nj-1}^{+}$$

We obtain a system of $2n$ algebraic equations (for $0 \leqslant \gamma < 1$) to determine the $2n$ unknowns in (4.3)

$$(G_{mj}^{-} + G_{Nj-1}^{+} + G_n) \sqrt{G_{Nj}^{+}} = (G_{mj}^{-} + \gamma G_{Nj-1}^{+} + (1-\gamma) G_{Nj}^{+} + G_n) \sqrt{G_{Nj-1}^{+}}$$

$$\left(\frac{\partial \Phi}{\partial G_{mj}^{-}} = 0\right)$$

$$\frac{G_{mj+1}^{-} + G_n}{(G_{mj+1}^{-} + G_{Nj}^{+} + G_n)^2} + \frac{\gamma G_{Nj+1}^{+}}{(G_{mj+1}^{-} + \gamma G_{Nj}^{+} + (1-\gamma) G_{Nj+1}^{+} + G_n)^2} =$$

$$= \frac{G_{mj}^{-} + \gamma G_{Nj-1}^{+} + G_n}{(G_{mj}^{-} + \gamma G_{Nj-1}^{+} + (1-\gamma) G_{Nj}^{+} + G_n)^2} \qquad \left(\frac{\partial \Phi}{\partial G_{Nj}^{+}} = 0\right)$$

$$\frac{G_{m1}^{-} + G_n}{(G_{m1}^{-} + G_{N0}^{+} + G_n)^2} + \frac{\gamma G_{N1}^{+}}{(G_{m1}^{-} + \gamma G_{N0}^{+} + (1-\gamma) G_{N1}^{+} + G_n)^2} = 1 \qquad \left(\frac{\partial \Phi}{\partial G_{N0}^{+}} = 0\right)$$

$$G_{m0}^{-} + G_{N0}^{+} = 1 - G_n \qquad (j = 1, \ldots, n-1)$$

In the $\gamma = 1$ limiting case the corresponding equations have the simple solution

$$G_{m0}^{-} = G_{m0} = 1 - G_n^{1/(n+1)}, \qquad G_{mj}^{-} = 0 \qquad (j = 1, \ldots, n-1)$$
$$G_{Nj}^{+} = G_n^{(j+1)/(n+1)} - G_n \qquad (j = 0, 1, \ldots, n-1; \gamma = 1) \tag{4.4}$$

Substituting (4.4) into (4.2) we obtain the relation between the maximum useful load G_n and the magnitude of the functional Φ

$$\Phi = (n+1)(1 - G_n^{1/(n+1)}) + G_n - 1 \quad \text{for } \gamma = 1 \tag{4.5}$$

As $n \to \infty$ this formula transforms into (3.14). Relation (4.5) is shown in Fig. 5 as the dependence $G_n(\Phi)$ for $n = 1, 2, 4, \infty$. It is seen from the figure that the passage from $n = 1$ to $n = 2$ realizes approximately one-third of the greatest possible gain in useful load and the passage to $n = 4$ realizes approximately two-thirds. The remaining third is realized by the passage from $n = 4$ to $n = \infty$. In the $\gamma = 0$ case analyzed in [2], the overwhelming part of the gain is realized for $n = 2$. It is interesting to note that the stepwise decrease in the weight of the power source shifts the upper bound of the range of admissible values of the functional Φ to $\Phi = n$ ($G_n = 0$ for $\Phi = n$).

5. Appendix 1. The determination of the optimum discharges q_m and q_ν is made from the condition of an absolute maximum of the function H^*, at each instant, in the independent variables $0 \leqslant q_m < \infty$, $0 \leqslant q_\nu < \infty$.

The momenta p_m and p_ν in expression (2.4) for H^* must be strictly greater than zero along the optimum solution

$$p_m(t) > 0, \qquad p_\nu(t) > 0 \tag{5.1}$$

Otherwise, from the condition of maximum H^* would follow either $q_m = \infty$ or $q_\nu = \infty$ (or both). All this contradicts the condition of finite discharges.

Let us introduce the total discharge

$$q = q_m + \gamma q_\nu \; (0 \leqslant q < \infty, 0 \leqslant q_m < q, 0 \leqslant q_\nu \leqslant q / \gamma) \tag{5.2}$$

and let us write the function H^* thus by using it

$$H^* = - p_m q_m + |\mathbf{p}_v| \sqrt{(2g / \alpha) G_N q} / (G_m + G_N + G_n) - (p_\nu - \gamma p_m) q_\nu \tag{5.3}$$

Hence, it is seen that the optimum distribution of the total discharge q between the components q_m and q_ν is determined by the sign of the combination $\Delta = - p_\nu - \gamma p_m$.

Indeed, if $\Delta > 0$, then the maximum of H^* in q_ν is achieved at $q_\nu = 0$; and if $\Delta < 0$, for $q_\nu = q/\gamma$, i.e. for $q_m = 0$. In the $\Delta = 0$ case, the function H^* is independent of each of the discharges q_m and q_ν separately but is determined by the total discharge q. Hence, in each of the three possible cases the function H^* depends only on one of the controls q_m, q_ν or q

$$H_1^* = - p_m q_m + |\mathbf{p}_v| \sqrt{(2g / \alpha) G_N q_m} / (G_m + G_N + G_n) \quad \text{for } \Delta > 0 \; (q_\nu = 0)$$
$$H_2^* = - p_\nu q_\nu + |\mathbf{p}_v| \sqrt{(2g / \alpha) G_N \gamma q_\nu} / (G_m + G_N + G_n) \quad \text{for } \Delta < 0 \; (q_m = 0)$$
$$H_3^* = - p_m q + |\mathbf{p}_v| \sqrt{(2g / \alpha) G_N q} / (G_m + G_N + G_n) \quad \text{for } \Delta = 0 \tag{5.4}$$

From the stationarity conditions $\partial H_1^*/\partial q_m = 0$ and $\partial H_2^*/\partial q_\nu = 0$, which here correspond to the condition of the absolute maximum, we obtain the necessary relations (2.6) which determine the optimum controls q_m and q_ν for $\Delta > 0$ and for $\Delta < 0$.

For $\Delta = 0$ there follows from the condition $\partial H_3/\partial q = 0$ that

$$q_m + \gamma q_\nu = \frac{(g / \alpha) G_N p_v^2}{2 p_m^2 (G_m + G_N + G_n)^2} \quad \text{for } \Delta = 0 \tag{5.5}$$

When Δ vanishes in a certain finite segment rather than at isolated points (and precisely such a case will actually be singular), then the derivative $\dot{\Delta}$ will also be zero in this interval

$$\dot{\Delta} = \dot{p}_v - \gamma\dot{p}_m = \frac{1}{2}\dot{p}_m\left(1 - 2\gamma - \frac{G_m + G_n}{G_N}\right) = 0 \tag{5.6}$$

Let us investigate the two possible variants.

First $\dot{p}_m = 0$. Using (5.5) we eliminate $(q_m + \gamma q_v)$ from (3.5)

$$\dot{p}_m = \frac{(g/\alpha)G_N}{(G_m + G_N + G_n)^3}\frac{p_v^2}{p_m^2} \tag{5.7}$$

For this expression to vanish it is necessary that either $G_N = 0$ or $\mathbf{p}_v = 0$. In either case, the motion will be inertial without thrust since the discharges q_m and q_v will vanish according to (2.6) and (5.5). Hence, both situations will be retained until the end of the motion. For $G_N = 0$ this requires no explanation but the momentum $\mathbf{p}_v$, as follows from (2.1)

$$\mathbf{p}_v = -\mathbf{p}_r, \qquad \mathbf{p}_r = -\mathbf{p}_v \ \partial\mathbf{R} / \partial\mathbf{r} \tag{5.8}$$

can vanish only identically along all trajectories. This case is degenerate since the motion will be entirely passive.

Hence, vanishing of the function Δ because $\dot{p}_m$ equals zero is only possible at the end of controlled motion, when G_N vanishes, i.e. the derivative $\dot{p}_m$ can be considered positive everywhere.

The second variant remains

$$1 - 2\gamma - (G_m + G_n) / G_N = 0 \tag{5.9}$$

This expression denotes the proportional change in the weight of the power source G_N and the store of working material G_m. In combination with (5.5), it yields (2.8) which determines the optimum controls q_m and q_v at $\Delta = 0$.

6. Appendix 2. Alternation of the extrema (3.1) to (3.3) is determined by the sign of the function $\Delta(t)$ from (2.5). Let us first consider the intermediate case $0 < \gamma < 1$. It follows from the optimum boundary conditions for the momenta (3.4) and (3.6) that the initial value of the function Δ is greater than zero and the final value is less than zero ($\Delta(0) > 0$, $\Delta(T) < 0$). It is necessary to investigate the nature of the behavior of this function within the interval $[0, T]$.

The time derivative of the function Δ is defined by (5.6). As is shown in the preceding section, the derivative $\dot{p}_m$ is positive.

Therefore, the change in the sign of the function Δ depends on the combination

$$\chi = 1 - 2\gamma - (G_m + G_n) / G_N \tag{6.1}$$

At the beginning of the motion (3.4), $\Delta(0) > 0$; hence, a change in weight occurs according to (3.1). According to (3.1), the weight G_N remains constant but G_m decreases, i.e. along (3.1) χ increases. If it

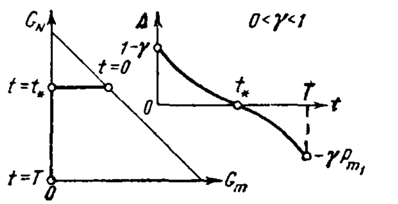

Fig. 6.

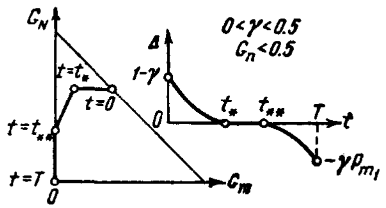

Fig. 7.

is assumed that $\dot{\Delta}(0) \geqslant 0$, then the final value $\Delta(T)$ will be positive because $\Delta(0) > 0$ and $\dot{\Delta}(t) \geqslant 0$ (because of the growth of χ). Positiveness of $\Delta(T)$ contradicts condition (3.6).

Hence, the derivative of the function Δ is negative at the beginning of the motion. Let us investigate two possible situations which can occur later.

1. The derivative $\dot{\Delta}(t)$ remains negative everywhere. Then $\Delta(t)$ crosses the t-axis at a certain time $t = t_*$ (since $\Delta(0) > 0$ and $\Delta(T) < 0$). In the first interval $0 \leqslant t \leqslant t_*$ equation (3.1) will hold, according to which $G_N(t) \equiv G_{N0}$ and in the second $t_* \leqslant t \leqslant T$, equation (3.3) holds along which $G_m(t) \equiv G_m(t_*)$ (see Fig. 6), where $G_m(t_*) = 0$ since $G_m^*(T) = 0$ according to condition (1.5).

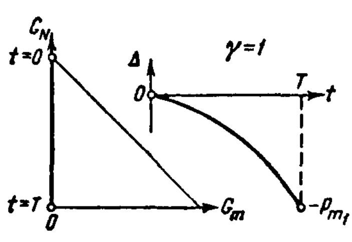

Fig. 8.

2. The derivative $\dot{\Delta}(t)$ vanishes at $\Delta(t) = 0$ ($t = t_*$). In the interval $0 \leqslant t \leqslant t_*$ the change in weight occurs according to equation (3.1). At $t = t_*$ it is replaced by (3.2). According to (3.2) the weights G_m and G_N change proportionately so that χ and $\dot{\Delta}$ remain zero. At a certain time $t = t_{**}$ the weight G_m vanishes. After this only G_N can change. The function Δ^* (meaning $\dot{\Delta}$ also) again becomes negative as G_N diminishes. The function Δ goes over into the negative domain and the control of the weight is accomplished according to equation (3.3) (see Fig. 7).

3. The derivative $\dot{\Delta}(t)$ vanishes for $\Delta(t) > 0$. In the domain $\Delta > 0$ the weights vary according to equation (3.1), according to which χ increases. Hence, after having vanished, the derivative $\dot{\Delta}$ becomes positive and the function Δ never falls into the domain of negative values, which contradicts condition (3.6).

4. The derivative $\dot{\Delta}(t)$ vanishes for $\Delta(t) < 0$. This possibility also drops out since the function χ must decrease in the domain $\Delta < 0$, according to (3.3) and (6.1). But for the function Δ to drop into the negative domain at a time $t = t_*$ it is necessary that $\dot{\Delta}(t_*) < 0$. Hence, in the domain $\Delta < 0$ the derivative $\dot{\Delta}$ remains negative.

Hence, the latter two variants 3 and 4 drop out. There remains to examine the limiting cases $\gamma = 1$ and $\gamma = 0$.

For $\gamma = 1$ the initial value of the function Δ is zero (3.4) and its derivative (5.6)

$$\dot{\Delta} = -\frac{1}{2} p_m \left(1 + \frac{G_m + G_n}{G_N}\right) \quad \text{for } \gamma = 1 \qquad (6.2)$$

is negative everywhere. Hence, all the subsequent values of Δ lie in the negative domain and only the weight of the power source G_N changes during the motion according to equation (3.3). The weight of the working material G_m is here identically zero since $G_m(T) = 0$ (Fig. 8) according to condition (1.4).

In the $\gamma = 0$ case, the initial value of the function Δ is one and the final value is zero (3.4) and (3.6). The function Δ cannot fall into the negative domain since equation (3.3) has no meaning for $\gamma = 0$. Hence, Δ vanishes either at the end of the motion and then $G_N(t) \equiv G_{N0}$ (Fig. 9) or at a certain time $t = t_* < T$. Then the derivative $\dot{\Delta}(t_*)$

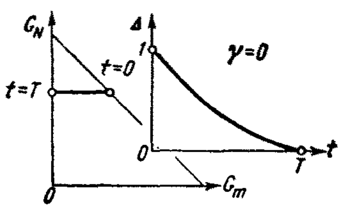

Fig. 9.

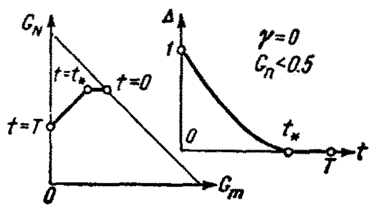

Fig. 10.

also vanishes and remains zero to the end of the motion. On the first section $0 \leqslant t \leqslant t_*$ the change in weight occurs according to (3.1), on the second section $t_* \leqslant t \leqslant T$ according to (3.2) (Fig. 10). The remaining possibilities are eliminated exactly as was done for $0 < \gamma < 1$.

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